Lecture 1 - Vectors

A Puzzle...

The sum starts at 0. Players alternate by choosing an integer from 1 to 10 and then adding it to the sum. The player who gets to 100 wins. You go first. What is the winning strategy?

Solution
Choose 1 (and then respond to your opponents number x with $11 - x$)

Introduction

Welcome to Ph 1a Section 5 (the coolest section of them all)!

TA's Information

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- Ask questions!

"The goal of teaching should not be to implant in the student's mind every fact that the teacher knows now; but rather to implant a way of thinking that will enable the student, in the future, to learn in one year what the teacher learned in two years. Only in that way can we continue to advance from one generation to the next."
 - *A Backward Look to the Future*, E. T. Jaynes

Vectors: The quest begins!

Vectors

A <u>*vector*</u> is a defined as a direction in space with a given length. For example, here is a perfectly nice vector
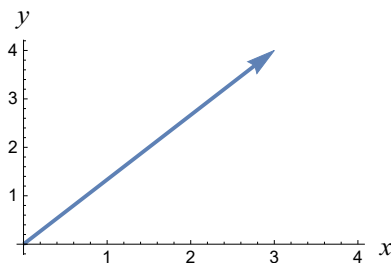
Notation
We will denote this vector as $\vec{A}$. In Cartesian coordinates, this vector would be written as

$$\vec{A} = 3\,\hat{x} + 4\,\hat{y} \tag{1}$$

where $\hat{x}$ and $\hat{y}$ represent the unit vectors in the x and y directions, respectively. If we were in 3D coordinates, $\hat{z}$ would represent a unit vector in the z direction. Other common notations for the components include

$$\vec{A} = (3, 4) \tag{2}$$
$$\vec{A} = \langle 3, 4 \rangle \tag{3}$$

When using such notations, we implicitly assume Cartesian coordinates (and not, for example, polar coordinates).

2D Vectors

A generic 2D vector in Cartesian coordinates is

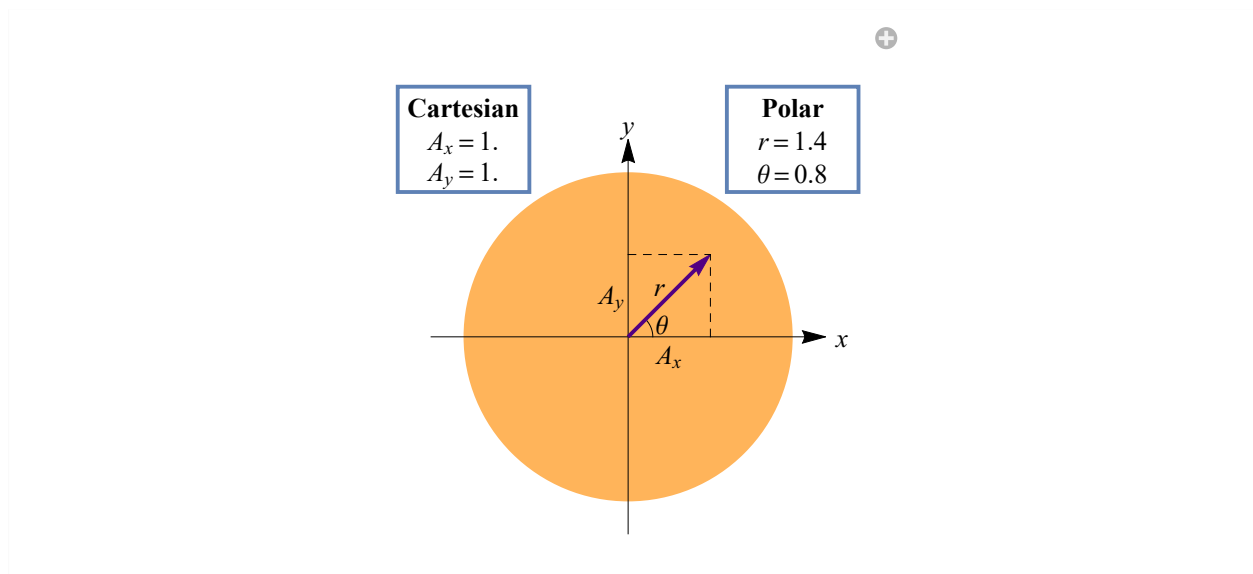$$\vec{A} = A_x\,\hat{x} + A_y\,\hat{y} \tag{4}$$

The *magnitude* of the vector $\vec{A}$ equals $\sqrt{A_x^2 + A_y^2}$ and is denoted by $|\vec{A}|$. The *zero vector* will be denoted by $\vec{0}$ and represents a vector with 0 for each of its Cartesian coordinates. In polar coordinates, $\vec{A}$ would be written as

$$\vec{A} = r\,\mathrm{Cos}[\theta]\,\hat{x} + r\,\mathrm{Sin}[\theta]\,\hat{y} \tag{5}$$

where

$$r = \sqrt{A_x^2 + A_y^2} \tag{6}$$
$$\theta = \mathrm{ArcTan}\left[\frac{A_y}{A_x}\right] \tag{7}$$

The Manipulate shown above is interactive, so that in Mathematica you can drag the purple arrow around and see A_x, A_y, r, and θ automatically update. Verify your understanding by moving the arrow in such a way that:

- A_x changes but A_y does not (and the converse)

- r changes but θ does not (and the converse)

Notation

The above notation $\mathrm{ArcTan}\left[\frac{A_y}{A_x}\right]$ may appear confusing because the "A" and "T" are capitalized and I use hard brackets "[⋯]" instead of parenthesis "(⋯)"; this is purely my own custom, since it follows *Mathematica* format (so you can enter it directly as *Mathematica* code). In other words, $\mathrm{ArcTan}\left[\frac{A_y}{A_x}\right]$ is the same thing as $\tan^{-1}\left(\frac{A_y}{A_x}\right)$ or

$\arctan\left(\frac{A_y}{A_x}\right)$. I will do this same capitalization and hard brackets format with any function: for example $\text{Cos}[\pi]$ is equivalent to $\cos(\pi)$.

Trig Review

Time for a quick math aside. There are certain functions that you should know and love, and which you will be using in every class you take as an undergraduate. Both sine (Sin) and cosine (Cos) fall into this category. The two functions are shown below.
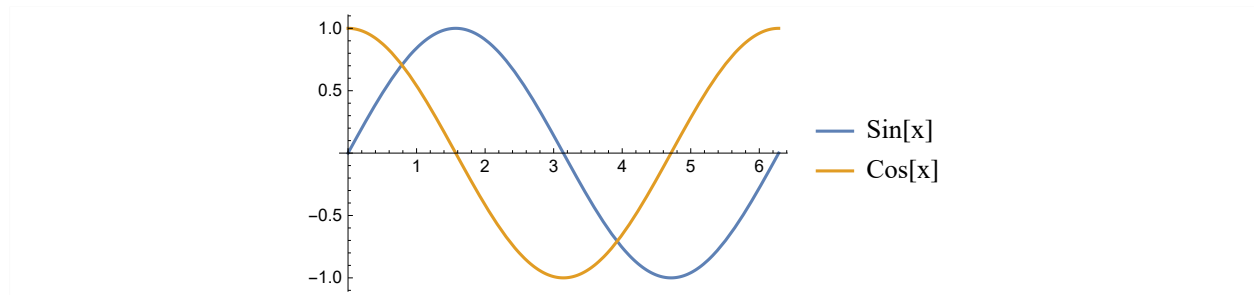
You should memorize the following facts:

$$\text{Cos}[0] = 1 \qquad \text{Sin}[0] = 0$$
$$\text{Cos}\left[\frac{\pi}{6}\right] = \frac{\sqrt{3}}{2} \qquad \text{Sin}\left[\frac{\pi}{6}\right] = \frac{1}{2}$$
$$\text{Cos}\left[\frac{\pi}{4}\right] = \frac{\sqrt{2}}{2} \qquad \text{Sin}\left[\frac{\pi}{4}\right] = \frac{\sqrt{2}}{2} \tag{8}$$
$$\text{Cos}\left[\frac{\pi}{3}\right] = \frac{1}{2} \qquad \text{Sin}\left[\frac{\pi}{3}\right] = \frac{\sqrt{3}}{2}$$
$$\text{Cos}\left[\frac{\pi}{2}\right] = 0 \qquad \text{Sin}\left[\frac{\pi}{2}\right] = 1$$

In addition, these properties of Sin and Cos will come up occasionally:

$$\text{Sin}[x + y] = \text{Sin}[x]\,\text{Cos}[y] + \text{Cos}[x]\,\text{Sin}[y]$$
$$\text{Cos}[x + y] = \text{Cos}[x]\,\text{Cos}[y] - \text{Sin}[x]\,\text{Sin}[y] \tag{9}$$
$$\text{Sin}[2\,x] = 2\,\text{Sin}[x]\,\text{Cos}[x]$$
$$\text{Cos}[2\,x] = \text{Cos}[x]^2 - \text{Sin}[x]^2 = 2\,\text{Cos}[x]^2 - 1 = 1 - 2\,\text{Sin}[x]^2 \tag{10}$$
$$\text{Sin}[\pi - x] = \text{Sin}[x]$$
$$\text{Cos}[\pi - x] = -\text{Cos}[x] \tag{11}$$

Lastly, you need to know the Taylor series of both functions. As a reminder, the Taylor series of a function $f[x]$ about the point $x = 0$ is given by

$$f[x] = f[0] + \frac{f'[0]}{1!}\,x + \frac{f''[0]}{2!}\,x^2 + \frac{f'''[0]}{3!}\,x^3 + \cdots \tag{12}$$

The Taylor series of the Sin and Cos functions are

$$\text{Sin}[x] = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!} = \frac{e^{ix} - e^{-ix}}{2i}$$
$$\text{Cos}[x] = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j}}{(2j)!} = \frac{e^{ix} + e^{-ix}}{2} \tag{13}$$

where the last equality uses Euler's formula

$$e^{ix} = \text{Cos}[x] + i\,\text{Sin}[x] \tag{14}$$

A final, extremely useful, Taylor series that you should know is that of the exponential function

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{j=0}^{\infty} \frac{x^j}{j!} \tag{15}$$
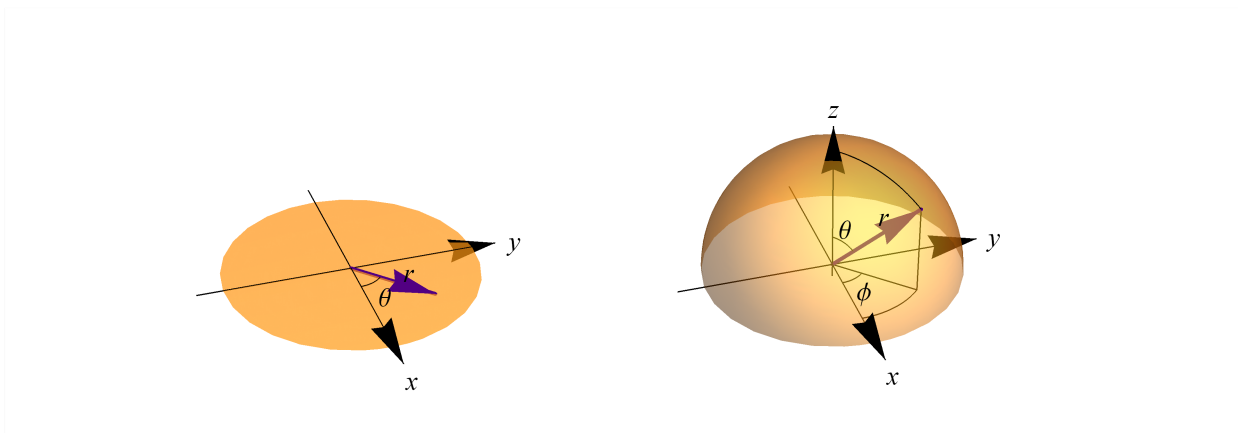
3D Vectors

A generic 3D vector in Cartesian coordinates is

$$\vec{A} = A_x\,\hat{x} + A_y\,\hat{y} + A_z\,\hat{z} \tag{16}$$

which in spherical polar coordinates equals

$$\vec{A} = r\,\text{Sin}[\theta]\,\text{Cos}[\phi]\,\hat{x} + r\,\text{Sin}[\theta]\,\text{Sin}[\phi]\,\hat{y} + r\,\text{Cos}[\theta]\,\hat{z} \tag{17}$$

where $\phi \in [0, 2\pi)$ is the polar angle and $\theta \in [0, \pi]$ is the azimuthal angle.

Unfortunately, as seen above, physicists take the horrible convention where θ is the polar angle in 2D radial coordinates but ϕ is the polar angle in 3D spherical coordinates. Therefore, to regain polar coordinates from spherical coordinates you must set $\theta = \frac{\pi}{2}$ and then rename $\phi = \theta$. (Mathematicians, on the other hand, tend to switch the definitions of θ and ϕ in spherical coordinates to avoid this issue!)

We can convert from Cartesian coordinates to spherical coordinates using

$$r = \sqrt{A_x^2 + A_y^2 + A_z^2} \tag{18}$$

$$\phi = \text{ArcTan}\left[\frac{A_y}{A_x}\right] \tag{19}$$

$$\theta = \text{ArcTan}\left[\frac{\left(A_x^2 + A_y^2\right)^{1/2}}{A_z}\right] = \text{ArcCos}\left[\frac{A_z}{\left(A_x^2 + A_y^2 + A_z^2\right)^{1/2}}\right] \tag{20}$$

(I encourage you to verify that both of these expressions for θ are equal!)

The last 3D coordinate system worth mentioning is cylindrical coordinates. This system is rarely used in classical mechanics and the book does not mention it, so we will skip it for now. However, you invited to consult Wikipedia to refresh your memory on this system (this system will definitely be used next term in Electromagnetism).

Including Cartesian coordinates, here is a pictorial summary of the four most important coordinate systems.
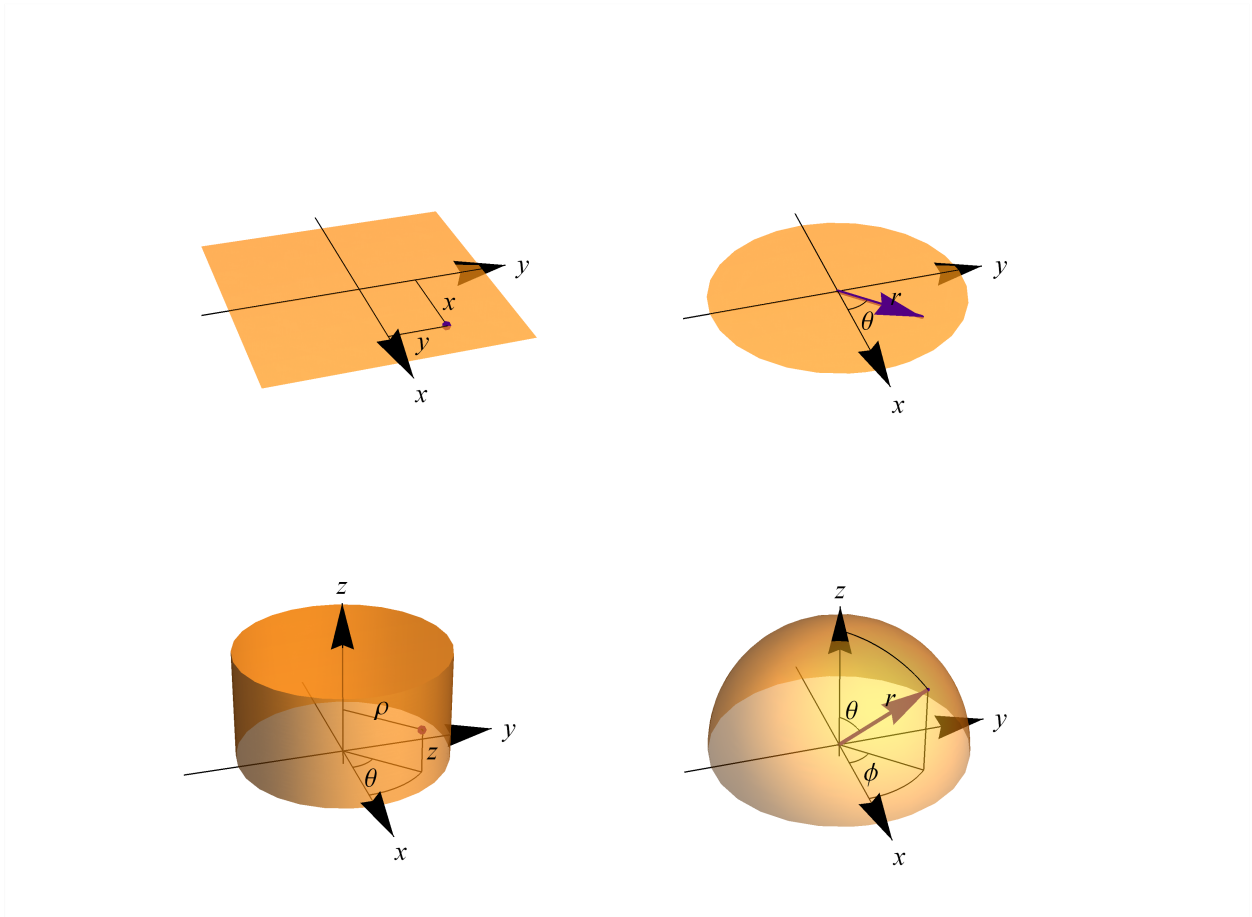
Example

Convert the three points with Cartesian coordinates $p_1 = (1, 0, 0)$, $p_2 = (1, 1, 0)$, and $p_3 = (0, 0, 1)$ into cylindrical and spherical coordinates.

Solution

We start with spherical coordinates. The three points are sufficiently nice (either falling on an axis or symmetrically between two axes) that you can infer r, θ, and ϕ by eye using the diagram above. Then you can double check your work by using Equations (18)-(20). In either case, you should obtain

$$
\begin{array}{cccc}
 & r & \theta & \phi \\
p_1 & 1 & \frac{\pi}{2} & 0 \\
p_2 & \sqrt{2} & \frac{\pi}{2} & \frac{\pi}{4} \\
p_3 & 1 & 0 & -
\end{array}
\tag{21}
$$

Note that the ϕ value of p_3 can be anything, since spherical coordinates do not have a unique representation at the poles.

Cylindrical coordinates follow the same Equations (6)-(7) as polar coordinates, with the z-component unchanged. Thus the three points become

$$
\begin{array}{cccc}
 & \rho & \theta & z \\
p_1 & 1 & 0 & 0 \\
p_2 & \sqrt{2} & \frac{\pi}{4} & 0 \\
p_3 & 0 & - & 1
\end{array}
\tag{22}
$$

Note that the θ coordinate can be anything for p_3, since cylindrical coordinates do not have a unique representation on the z-axis. □
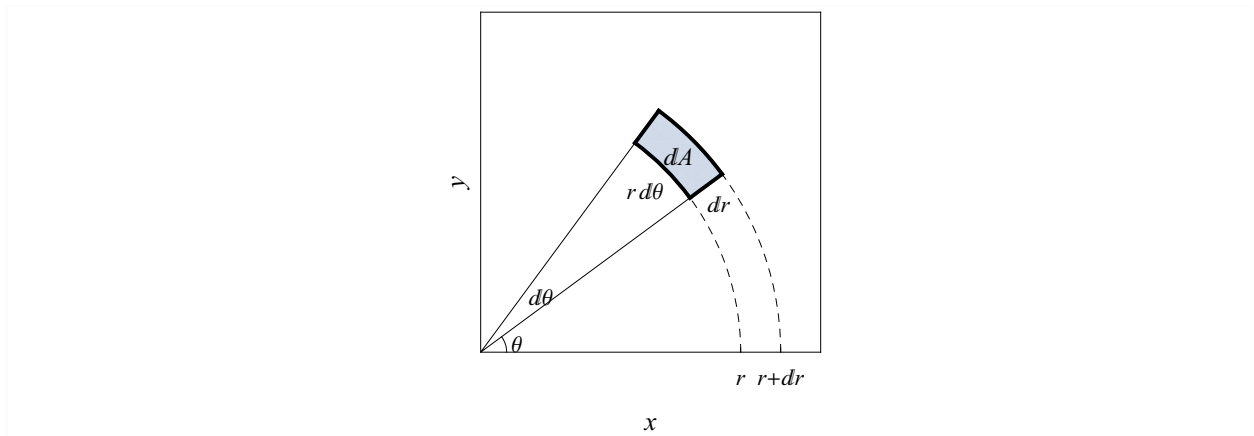
Integration

When doing 2D integrals in Cartesian coordinates, we are familiar with the area element $dx\,dy$. We use this when computing an integral $\int\int f[x, y]\,dx\,dy$. But where does $dx\,dy$ come from? If we consider a generic point (x, y) and infinitesimally displace it in the x-direction and y-direction by dx and dy, respectively, we find the points (x, y), $(x + dx, y)$, $(x, y + dy)$, and $(x + dx, y + dy)$. What is the area of the region encompassed by these points? Simply $dx\,dy$!

<div align="center">Cartesian Coordinate Area Element $dA = dx\,dy$</div>　(23)
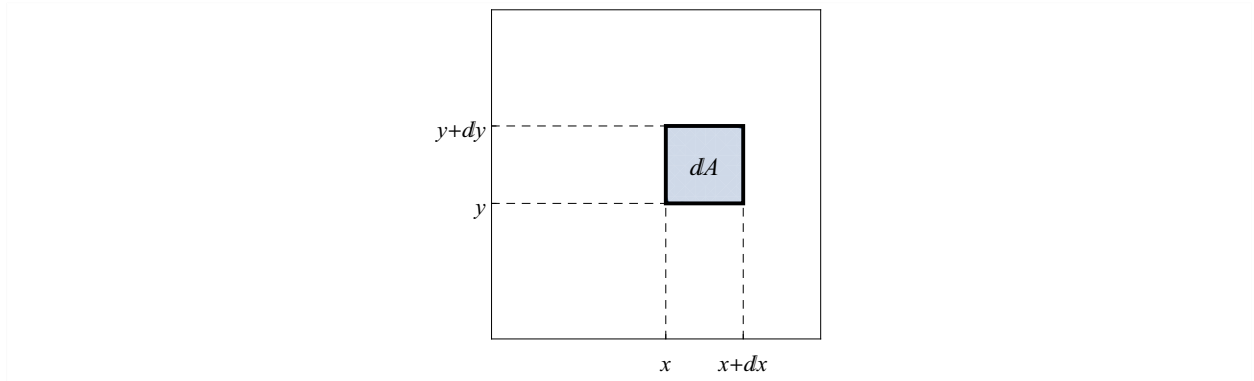
The situation is the same for polar coordinates, although perhaps slightly less familiar. Consider a generic point (r, θ) and the infinitesimally displaced points found when we vary the first coordinate by dr and the second coordinate by $d\theta$. Thus we have four points (r, θ), $(r + dr, \theta)$, $(r, \theta + d\theta)$, and $(r + dr, \theta + d\theta)$. As shown in the figure below, the element of this infinitesimal area equals $(dr)(r\,d\theta) = r\,dr\,d\theta$. Thus,

<div align="center">Polar Coordinate Area Element $dA = r\,dr\,d\theta$</div>　(24)

What does this imply? If we integrate $f[r, \theta]$ over a region in polar coordinates, then the integral must take the form

$$\int\int f[r, \theta]\,r\,dr\,d\theta$$　(25)

<u>Example</u> *(A Classic!)*
Prove that the area of a circle with radius R is πR^2 using both polar and Cartesian coordinates.

Solution

This problem is extremely simple in polar coordinates which, after all, are specifically designed to utilize rotational symmetry. To find an area of an object, we use Equation (25) with the integrand $f[r, \theta] = 1$. The bounds take the simple form

$$
\begin{aligned}
\text{Circle Area} &= \int_0^{2\pi} \int_0^R r \, dr \, d\theta \\
&= \int_0^{2\pi} \tfrac{1}{2} R^2 \, d\theta \\
&= \pi R^2
\end{aligned}
\tag{26}
$$

In Cartesian coordinates, this problem is both difficult to set up (because the integration bounds are awkward) and difficult to evaluate. Nevertheless, what doesn't kill you makes you stronger.

$$
\begin{aligned}
\text{Circle Area} &= \int_{-R}^{R} \int_{-(R^2-y^2)^{1/2}}^{(R^2-y^2)^{1/2}} dx \, dy \\
&= \int_{-R}^{R} 2 \, (R^2 - y^2)^{1/2} \, dy \\
&= \left(y \, (R^2 - y^2)^{1/2} + R^2 \, \text{ArcTan}\left[\frac{y}{(R^2 - y^2)^{1/2}} \right] \right)\Big|_{y=-R}^{y=R} \\
&= \tfrac{\pi}{2} R^2 - \left(-\tfrac{\pi}{2} R^2 \right) \\
&= \pi R^2
\end{aligned}
\tag{27}
$$

As you can see, it greatly behooves you to use polar coordinates in problems that have spherical symmetry. □

Vector Addition and Scalar Multiplication

Unless otherwise stated, we will now work in 2D, but every statement we will be considering naturally extends to 3D.

In Cartesian coordinates, *vector addition* is straightforward addition in every coordinate. Given two vectors $\vec{A} = A_x \hat{x} + A_y \hat{y}$ and $\vec{B} = B_x \hat{x} + B_y \hat{y}$, we can add the two together as

$$
\vec{A} + \vec{B} \equiv (A_x + B_x) \hat{x} + (A_y + B_y) \hat{y}
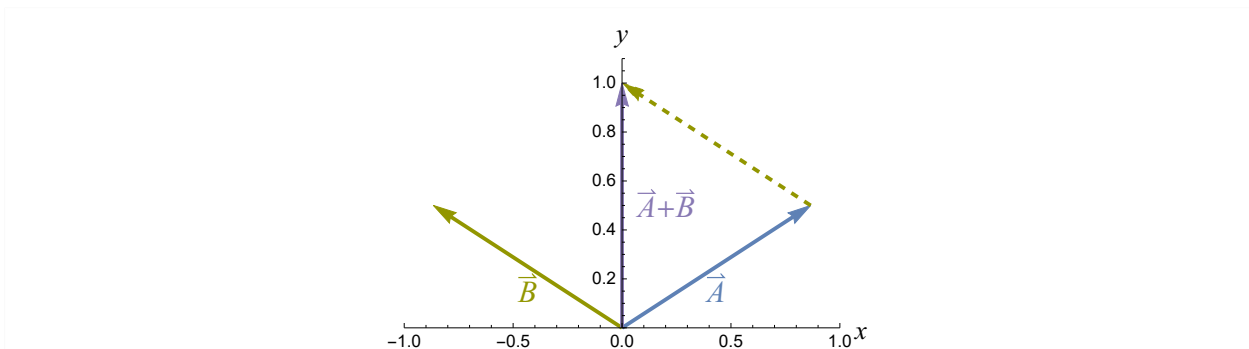\tag{28}
$$

Notation

I use $\equiv$ to denote a definition.

Scalar multiplication of a vector $\vec{A} = A_x \hat{x} + A_y \hat{y}$ by a scalar c is defined as

$$
c \vec{A} \equiv c A_x \hat{x} + c A_y \hat{y}
\tag{29}
$$

Example

Let $\vec{A}$ be the unit vector with polar angle $\theta = \frac{\pi}{6}$ and $\vec{B}$ be the unit vector with polar angle $\theta = \frac{5\pi}{6}$. What is $\vec{A} + \vec{B}$?

Solution

In Cartesian coordinates, $\vec{A} = \text{Cos}\left[\frac{\pi}{6}\right]\hat{x} + \text{Sin}\left[\frac{\pi}{6}\right]\hat{y} = \frac{\sqrt{3}}{2}\hat{x} + \frac{1}{2}\hat{y}$ and $\vec{B} = -\frac{\sqrt{3}}{2}\hat{x} + \frac{1}{2}\hat{y}$. Performing the vector

addition yields $\vec{A} + \vec{B} = \hat{y}$. Of course, the symmetry of the problem indicated that $\vec{A} + \vec{B}$ must point in the *y*-direction and have no *x*-component. □

Dot Product

Given two vectors $\vec{A} = A_x\,\hat{x} + A_y\,\hat{y}$ and $\vec{B} = B_x\,\hat{x} + B_y\,\hat{y}$, we define the *dot product* of $\vec{A}$ and $\vec{B}$ as
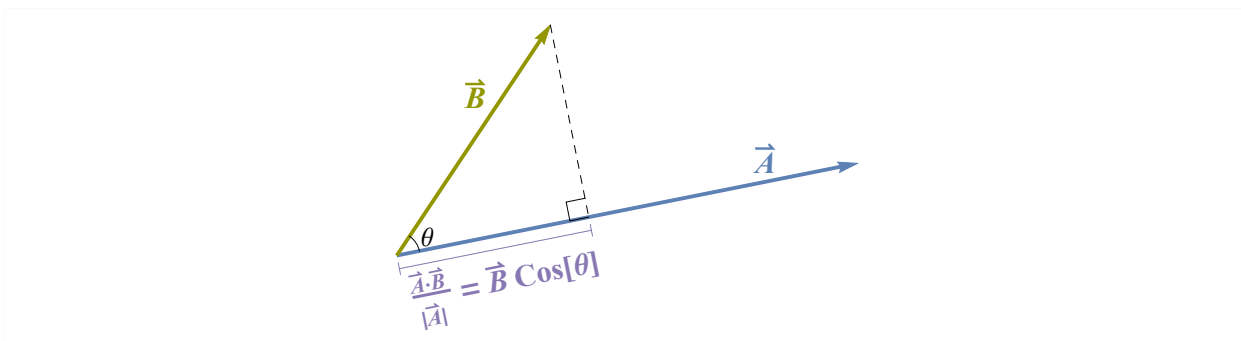
$$\vec{A} \cdot \vec{B} \equiv A_x\,B_x + A_y\,B_y \tag{30}$$

But a much better representation of the dot product (and the way you should visualize it in your mind) is using the following theorem.

Theorem

Given two vectors $\vec{A} = A_x\,\hat{x} + A_y\,\hat{y}$ and $\vec{B} = B_x\,\hat{x} + B_y\,\hat{y}$, the dot product of these two vectors equals

$$\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}|\,\text{Cos}[\theta] \tag{31}$$

Proof

We can write $\vec{A}$ and $\vec{B}$ in polar coordinates as (r_1, θ_1) and (r_2, θ_2), respectively. Then the dot product definition gives

$$\begin{aligned}
\vec{A} \cdot \vec{B} &= A_x\,B_x + A_y\,B_y \\
&= r_1\,r_2(\text{Cos}[\theta_1]\,\text{Cos}[\theta_2] + \text{Sin}[\theta_1]\,\text{Sin}[\theta_2]) \\
&= r_1\,r_2\,\text{Cos}[\theta_1 - \theta_2] \\
&= |\vec{A}|\,|\vec{B}|\,\text{Cos}[\theta]
\end{aligned} \tag{32}$$

where $\theta = |\theta_1 - \theta_2|$ is the angle between $\vec{A}$ and $\vec{B}$ (recall that Cosine is an even function, so it does not matter if $\theta_1 \le \theta_2$ or $\theta_2 \le \theta_1$). □

Properties of the Dot Product

Equation (31) is important! It immediately leads us to the most important way to think about the dot product:

The dot product of a vector $\vec{A}$ with *any* unit vector equals the component of $\vec{A}$ in the direction of that unit vector. $\tag{33}$
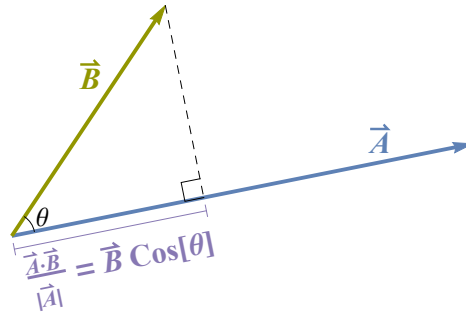
We will use several other useful properties of the dot product, namely, that the dot product is:

- A scalar

- Commutative ($\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$)

- Distributive (Any vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ satisfy $\vec{A} \cdot \left(\vec{B} + \vec{C}\right) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C}$)

- $\vec{A} \cdot \vec{A} = |\vec{A}|^2$

- If $\vec{A} \cdot \vec{A} = 0$, then $\vec{A} = \vec{0}$

- Note that all of these properties also hold in 3D Cartesian coordinates

Example
Consider the generic 2D vector $\vec{A} = A_x \hat{x} + A_y \hat{y}$. By taking the dot product of $\vec{A}$ with the appropriate unit vector, determine:

- The length of $\vec{A}$ projected along the *x*-axis

- The length of $\vec{A}$ projected along the $y = x$ axis
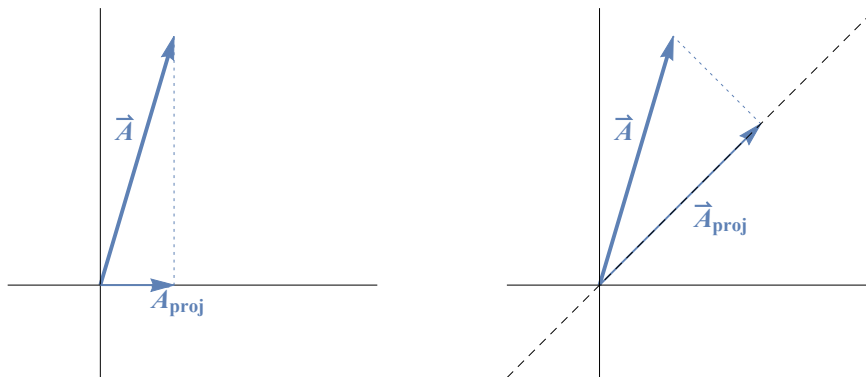
Solution
The length of $\vec{A}$ along the *x*-axis is given by

$$\begin{aligned}
\vec{A} \cdot \hat{x} &= (A_x \hat{x} + A_y \hat{y}) \cdot \hat{x} \\
&= A_x \hat{x} \cdot \hat{x} + A_y \hat{y} \cdot \hat{x} \\
&= A_x
\end{aligned} \tag{34}$$

as expected. The length of $\vec{A}$ along the $y = x$ axis (which is represented by the unit vector $\hat{n} = \frac{1}{\sqrt{2}} \hat{x} + \frac{1}{\sqrt{2}} \hat{y}$) is

given by

$$\vec{A} \cdot \hat{n} = (A_x \hat{x} + A_y \hat{y}) \cdot \left(\frac{1}{\sqrt{2}} \hat{x} + \frac{1}{\sqrt{2}} \hat{y} \right)$$

$$= A_x \hat{x} \cdot \frac{1}{\sqrt{2}} \hat{x} + A_x \hat{x} \cdot \frac{1}{\sqrt{2}} \hat{y} + A_y \hat{y} \cdot \frac{1}{\sqrt{2}} \hat{x} + A_y \hat{y} \cdot \frac{1}{\sqrt{2}} \hat{y} \qquad (35)$$

$$= \frac{A_x + A_y}{\sqrt{2}}$$

Note that the answer is symmetric with respect to A_x and A_y, as it must be because reflecting a vector about the $y = x$ axis (which amounts to sending $\vec{A} \to A_y \hat{x} + A_x \hat{y}$) should not change its projection along the $y = x$ axis. In addition, this answer makes sense in the limits when $A_x = 0$ or $A_y = 0$ (when $\vec{A}$ forms a 45°/45°/90° right triangle) as well as when $A_x = A_y$ (where $\vec{A} \cdot \hat{n} = |\vec{A}| = A_x \sqrt{2} = A_y \sqrt{2}$). □

The following Manipulate allows you to drag the two arrows around and see how the dot product changes. Utilizing this plot and the equations above, convince yourself of the following facts about the dot product:

- For any two vectors, their dot product is maximum when the vectors point in the same direction; minimum (and equals the negative value of its maximum value) when the vectors point in opposite directions; and zero when the vectors are perpendicular to each other

- In our definition, $\theta \in (-\pi, \pi]$ is not the usual polar angle because we do not impose counter-clockwise orientation on θ. For example, if $\vec{A} = \hat{x}$ and $\vec{B} = -\hat{y}$ then $\theta = \frac{\pi}{2}$ while the usual polar angle would be $\theta = \frac{3\pi}{2}$.

  However, because $\text{Cos}[2\pi - \theta] = \text{Cos}[\theta]$, we could have defined θ to be the polar angle. However, once you get to 3D, and $\vec{A}$ and $\vec{B}$ are not longer restricted to the *x-y* plane, our definition of θ is much more sensible.
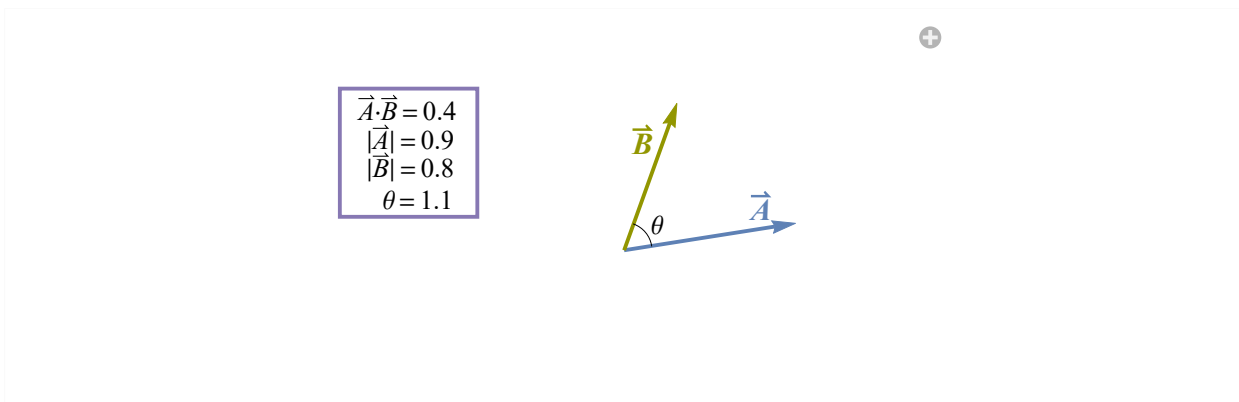
Example

Given a generic 2D vector $\vec{A} = A_x \hat{x} + A_y \hat{y}$, find the vector $\vec{B}$ with the same magnitude as $\vec{A}$ but which points in the direction of $\vec{A}$ rotated 90° counter-clockwise.
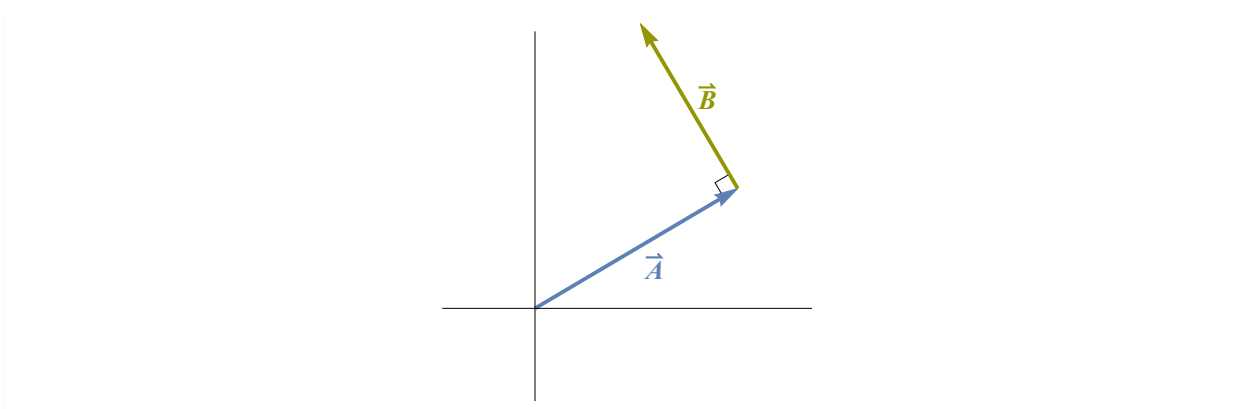
<u>Solution</u>

We illustrate two methods to solving this problem:

Method 1: Dot Product

We want to find a vector $\vec{B} = B_x\,\hat{x} + B_y\,\hat{y}$ satisfying $\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|\,\mathrm{Cos}\left[\frac{\pi}{2}\right] = 0$. Writing out the dot product explicitly in Cartesian coordinates,

$$A_x\,B_x + A_y\,B_y = 0 \tag{36}$$

which implies

$$B_y = -\frac{A_x}{A_y}\,B_x \tag{37}$$

Since $|\vec{B}| = |\vec{A}|$,

$$B_x^2 + B_y^2 = A_x^2 + A_y^2 \tag{38}$$

This implies that

$$B_x^2 + \left(\frac{A_x}{A_y}\right)^2 B_x^2 = A_x^2 + A_y^2 = A_y^2\left(1 + \frac{A_x^2}{A_y^2}\right) \tag{39}$$

or equivalently

$$\begin{aligned} B_x &= \pm A_y \\ B_y &= \mp A_x \end{aligned} \tag{40}$$

There are two vectors that satisfy this criteria, one that is rotated 90° clockwise from $\vec{A}$, and the other that is rotated 90° counterclockwise. By substituting in $A_y = 0$, we quickly see that we want to use the lower signs

$$\vec{B} = \langle -A_y,\, A_x \rangle \tag{41}$$

Method 2: Cross Product

Although we have not defined the cross product yet (see the next section), it provides a slick, alternative way to solve this problem. Think of $\vec{A}$ as a 3D vector with zero for its *z*-component, $\vec{A} = \langle A_x, A_y, 0\rangle$. We want to find a vector $\vec{B} = \langle B_x, B_y, 0\rangle$ which is perpendicular to $\vec{A}$, which implies that $\left|\vec{A}\times\vec{B}\right| = |\vec{A}|\,|\vec{B}|\,\mathrm{Sin}\left[\frac{\pi}{2}\right] = |\vec{A}|\,|\vec{B}|$. From the right-hand rule, or from Equation (7), $\vec{A}\times\vec{B}$ must point in the *z*-direction and hence have magnitude

$$\left|\vec{A}\times\vec{B}\right| = |A_x\,B_y - A_y\,B_x| \tag{42}$$

You can guess by inspection that this magnitude will be maximized if $B_y = A_x$ and $B_x = -A_y$ (or if both components take on the negative of these values), which you can prove more rigorously by letting $B_x = \left(A_x^2 + A_y^2\right)^{1/2}\mathrm{Cos}[\theta]$, $B_y = \left(A_x^2 + A_y^2\right)^{1/2}\mathrm{Sin}[\theta]$, substituting into the formula above, and differentiating with respect to θ (and checking that the extrema actually yield a maximum). This is done in the following Mathematica code.

```
Echo[#, "Possible B⃗: "] & /@ (Quiet@FullSimplify[(Ax^2 + Ay^2)^(1/2) {Cos[θ], Sin[θ]} /.
    Solve[D[Ax (Ax^2 + Ay^2)^(1/2) Sin[θ] - Ay (Ax^2 + Ay^2)^(1/2) Cos[θ], θ] == 0, θ][[2 ;; 3]], 0 < Ax && 0 < Ay]);
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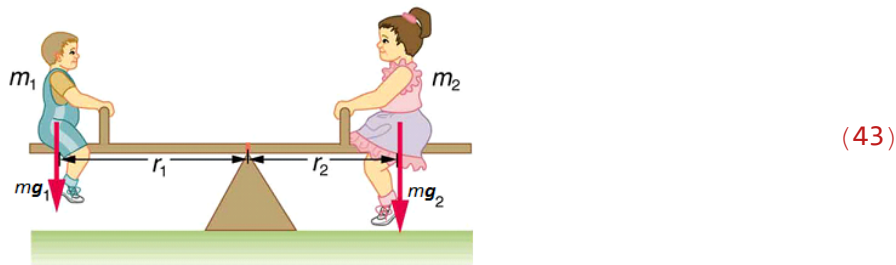
» Possible $\vec{B}$: {-Ay, Ax}

» Possible $\vec{B}$: {Ay, -Ax}

Therefore, we find as before the two possibilities $\vec{B} = \langle -A_y, A_x \rangle$ and $\vec{B} = \langle A_y, -A_x \rangle$, of which the former is the desired 90° counterclockwise rotation of $\vec{A}$. □

Cross Product

Intuitively, the cross product captures the idea that a bigger lever arm produces more torque. Keep this concept in the back of your mind as we work through the mathematics.

(43)

In this section, we work in 3D Cartesian coordinates because the cross product is not defined in 2D.

Given two vectors $\vec{A} = A_x\,\hat{x} + A_y\,\hat{y} + A_z\,\hat{z}$ and $\vec{B} = B_x\,\hat{x} + B_y\,\hat{y} + B_z\,\hat{z}$, we define the *cross product* of $\vec{A}$ and $\vec{B}$ as

$$\vec{A} \times \vec{B} \equiv (A_y\,B_z - A_z\,B_y)\,\hat{x} + (A_z\,B_x - A_x\,B_z)\,\hat{y} + (A_x\,B_y - A_y\,B_x)\,\hat{z} \tag{44}$$

A useful way to remember this formula is using the matrix determinant

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{45}$$

However, the best way to visualize the cross product in your head is through the following theorem.

<u>Theorem</u>
Given two vectors $\vec{A}$ and $\vec{B}$, the cross product $\vec{A} \times \vec{B}$ is a vector whose direction is given by the right-hand rule and whose magnitude equals

$$\left|\vec{A} \times \vec{B}\right| = |\vec{A}|\,|\vec{B}|\,\mathrm{Sin}[\theta] \tag{46}$$

where θ is the angle between $\vec{A}$ and $\vec{B}$.

Proof

This proof is straightforward but computational. Thus, before launching into the thick of things, we lay out the game plan. The proof will proceed in the following steps:

- We first compute $\left|\vec{A} \times \vec{B}\right|^2$ using Equation (44)

- We then compute $\left(\vec{A} \cdot \vec{B}\right)^2$ using the dot product definition $\vec{A} \cdot \vec{B} = A_x\,B_x + A_y\,B_y + A_z\,B_z$

- We will add these two results together to find that $\left|\vec{A} \times \vec{B}\right|^2 + \left(\vec{A} \cdot \vec{B}\right)^2 = |\vec{A}|^2\,|\vec{B}|^2$

- Finally, we will substitute the dot product relation $\vec{A} \cdot \vec{B} = |\vec{A}|\,|\vec{B}|\,\mathrm{Cos}[\theta]$ to obtain $\left|\vec{A} \times \vec{B}\right| = |\vec{A}|\,|\vec{B}|\,\mathrm{Sin}[\theta]$

<u>Proof</u>

Using Equation (44), we compute

$$\left|\vec{A}\times\vec{B}\right|^2 = (A_y B_z - A_z B_y)^2 + (A_z B_x - A_x B_z)^2 + (A_x B_y - A_y B_x)^2$$
$$= A_x^2(B_y^2 + B_z^2) + A_y^2(B_x^2 + B_z^2) + A_z^2(B_x^2 + B_y^2) \tag{47}$$
$$- 2\,(A_x A_y B_x B_y + A_y A_z B_y B_z + A_x A_z B_x B_z)$$

We next expand $\left(\vec{A}\cdot\vec{B}\right)^2 = (A_x B_x + A_y B_y + A_z B_z)^2$ to obtain

$$\left(\vec{A}\cdot\vec{B}\right)^2 = A_x^2 B_x^2 + A_y^2 B_y^2 + A_z^2 B_z^2$$
$$+ 2\,(A_x A_y B_x B_y + A_y A_z B_y B_z + A_x A_z B_x B_z) \tag{48}$$

Note that the last terms in both $\left|\vec{A}\times\vec{B}\right|^2$ and $\left(\vec{A}\cdot\vec{B}\right)^2$ are equal and opposite, so that they will cancel when we add both quantities,

$$\left|\vec{A}\times\vec{B}\right|^2 + \left(\vec{A}\cdot\vec{B}\right)^2 = A_x^2(B_x^2 + B_y^2 + B_z^2) + A_y^2(B_x^2 + B_y^2 + B_z^2) + A_z^2(B_x^2 + B_y^2 + B_z^2)$$
$$= (A_x^2 + A_y^2 + A_z^2)\,(B_x^2 + B_y^2 + B_z^2) \tag{49}$$
$$= |\vec{A}|^2\,|\vec{B}|^2$$

Using the dot product relation $\vec{A}\cdot\vec{B} = |\vec{A}|\,|\vec{B}|\,\mathrm{Cos}[\theta]$, we can rewrite this last formula as

$$\left|\vec{A}\times\vec{B}\right|^2 = |\vec{A}|^2\,|\vec{B}|^2 - \left(\vec{A}\cdot\vec{B}\right)^2$$
$$= |\vec{A}|^2\,|\vec{B}|^2 - |\vec{A}|^2\,|\vec{B}|^2\,\mathrm{Cos}[\theta]^2 \tag{50}$$
$$= |\vec{A}|^2\,|\vec{B}|^2\,\mathrm{Sin}[\theta]^2$$

which completes the proof of Equation (46). All that remains is to show that the vector $\vec{A}\times\vec{B}$ defined by Equation (44) always follows the right-hand rule. It might seem trivial, but proving this rigorously is surprisingly difficult, and I leave this fun problem as an exercise for the reader! □

Cross Product Properties

Equations (44) and (46) give us the keys to the cross product kingdom. From them, it is straightforward to extract that the cross product is:
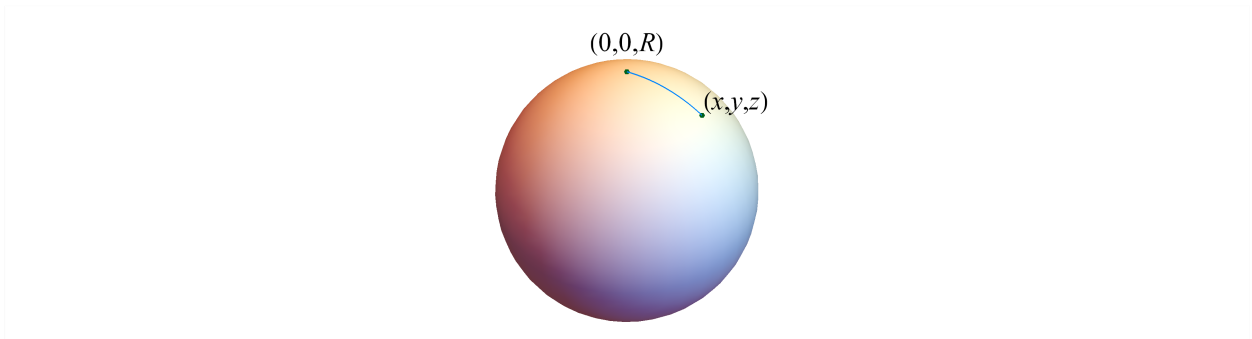
- A vector

- Anti-commutative $(\vec{A}\times\vec{B} = -\vec{B}\times\vec{A})$

- *Not* associative (Given vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, it is *not* true that $\vec{A}\times\left(\vec{B}\times\vec{C}\right) = \left(\vec{A}\times\vec{B}\right)\times\vec{C}$. Counter-example: $\vec{A} = \hat{x},\ \vec{B} = \hat{x},\ \vec{C} = \hat{y}$)

- Distributive over addition (Any vectors $\vec{A}$, $\vec{B}$, $\vec{C}$ satisfy $\vec{A}\times\left(\vec{B}+\vec{C}\right) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$)

- For any vectors $\vec{A}$, $\vec{B}$ and scalar c, $c\left(\vec{A}\times\vec{B}\right) = \left(c\,\vec{A}\right)\times\vec{B} = \vec{A}\times\left(c\,\vec{B}\right)$

- For any vector $\vec{A}$, $\vec{A}\times\vec{A} = \vec{0}$

- Given two vectors $\vec{A}$ and $\vec{B}$, the cross product $\vec{A}\times\vec{B}$ is orthogonal to both $\vec{A}$ and $\vec{B}$.
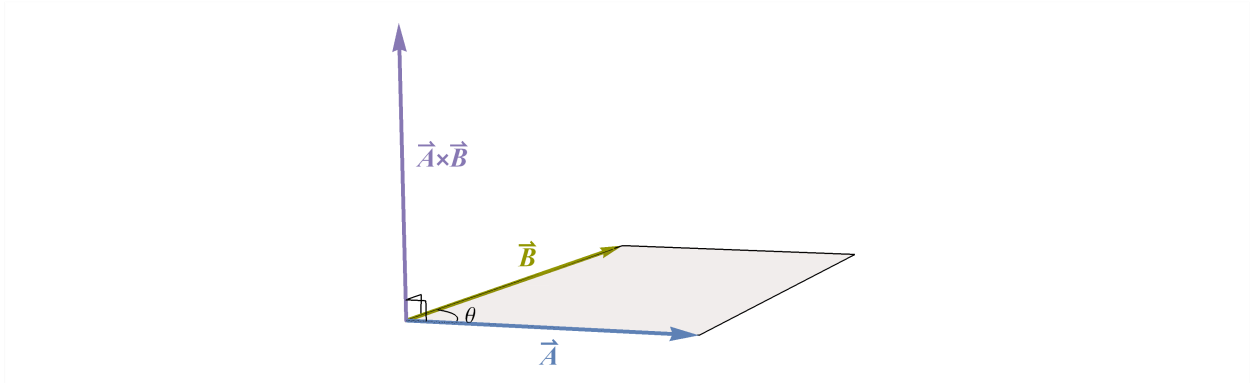
Math Bootcamps

There are two short but important math boot camps posted on my website that cover Coordinate Systems and Unit Vectors in more detail. If you cannot answer the following questions, you should read the corresponding boot camp:

- Coordinate Systems: Staying on a sphere of radius R, what is the shortest distance between the point $(0, 0, R)$ on the z-axis and a general point (x, y, z) on the sphere?

- Unit Vectors: Write the polar unit vectors $\hat{r}$ and $\hat{\theta}$ in terms of the Cartesian unit vectors $\hat{x}$ and $\hat{y}$.

Mathematica Initialization